Proceedings of the Institute of Mechanical Engineers, Vol. 201, No. C3, pp. 153-170, 1987.

[4]Kao, R., "Large Deformation Elastic-Plastic Buckling Analysis of Spherical Caps with Initial Imperfections," *Computers and Structures,* Vol. 11, 1980, pp. 609-619.

[5]Yang, T. Y. and Liaw, D. G., "Elastic-Plastic Dynamic Buckling of Thin-Shell Finite Elements with Asymmetric Imperfections," *AIAA Journal,* Vol. 26, No. 4, 1988, pp. 479-486.

[6]Hibbitt, H. D., Karlsson, B. I., and Sorensen, E. P., *ABAQUS—User Manual,* Version 4.5, HKS, Inc., RI, 1984.

[7]Bushnell, D., "BOSOR 5—Program for Buckling of an Elastic-Plastic Complex Shells of Revolution Including Large Deflections and Creep," *Computers and Structures,* Vol. 6, 1976, pp. 221-239.

[8]"FASOR—A Program for Stress, Buckling, and Vibration of Shells of Revolution," Advances in Engineering Software, Vol. 3, 1981, pp. 155-162.

[9]Bushnell, D., "Nonlinear Axisymmetric Behavior of Shells of Revolution," *AIAA Journal,* Vol. 5, No. 3, 1967, pp. 432-439.

Effect of Thermomechanical Coupling on the Response of Elastic Solids

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Introduction

THIS Note treats a condition that arises when a solid body is subjected to a highly transient heating or cooling, sometimes referred to as "thermal shock" exposure. The phenomenon is of greatest significance for completely elastic or "brittle" materials, where it may produce material fracture. Many high temperature materials such as ceramics and intermetallics fall into this category.

Thermal response of elastic solids is always accompanied by two important phenomena; the coupling between the thermal and mechanical energy and the inertia effects. A few problems have been solved to show their effects on the stress and the temperature distributions of isotropic solids subjected to thermal loads. However, one can investigate the reasonableness of neglecting these effects by comparing exact and approximate solutions to a specific problem. Some classical discussions of thermomechanical coupling include Mindlin and Goodman,[1] Boley,[2] and Soler and Brull.[3] Recently, Bahar and Hetnarski[4] extended their transfer matrix approach to deal with the coupled thermoelasticity of a layered medium. Additionally, Takeuti and Tsuju[5] considered the coupling effect when they analyzed the thermoelasticity problem of a plate heated by linear heat sources on both surfaces. They found that considering the coupling terms results in a delay in the progress of temperature distribution, a decrease in the value of the maximum temperature, and a narrower shape of the curve compared with neglecting the coupling effect. Further, there was a 9% increase in the thermal stresses for the case studied and a delay in the progress of the thermal stress distribution and the position of its maximum value with an increase in time. Other relevant recent studies include Takeuti and Furkawa[6] and Takeuti and Tanigawa.[7] The finite-element method was first applied to a coupled thermoelasticity problem by Chen and Ghoneim.[8] Results obtained[9] indicate that, under a high rate of loading, the coupling term should not be ignored as it could cause temperature fluctuations of the order of 5 to 10%.

The present Technical Note focuses on the effect of the thermomechanical coupling on the thermal response of an isotropic beam subjected to a nonuniform temperature. An analytical technique based on an eigenfunction method has been used to obtain the stress and the temperature distributions using the coupled thermoelasticity theory. Different comparisons have been made between the present solutions and those obtained by the classical uncoupled quasistatic thermoelasticity theory.

Analysis

Using energy principles, one can obtain the following equation for an elastic linear solid subjected to a temperature change:

$$K_{ij}T_{,ij} + \rho r = \rho c_e \dot{T} + T_0 c_{ijk1}\alpha_{k1}\dot{\epsilon}_{ij} \tag{1}$$

where ρ, c_e, T_0, r, c_{ijkl}, and α_{kl} are mass density, specific heat, reference temperature, heat supply per unit mass, the compliance tensor, and the expansion coefficient vector, and K_{ij}, T, and $\dot{\epsilon}_{ij}$ are the thermal conductivity tensor, the temperature, and the strain-rate tensor respectively. For isotropic materials, Eq. (1) takes the familiar form

$$kT_{,ii} + \rho r = \rho c_e \dot{T} + \alpha(3\lambda + 2\mu)T_0\dot{\epsilon}_{ij} \tag{2}$$

The appearance of the mechanical coupling term in the general thermoelastic problem can be interpreted as if an external action produces some variation of strains within the structural element. The equation shows that these strains are accompanied by variations in temperature and consequently by a flow of heat. Similarly, if the structure is subjected to a nonuniform temperature, the coupling term indicates that the way the structural element deforms due to this thermal exposure affects the temperature distribution.

Equation (2) and the following equilibrium equation compose the two governing differential equations for any thermal elastic problem

$$\sigma_{ij,j} + f_i = \rho \ddot{u}_i, \tag{3}$$

Where σ_{ij}, f, and u are the stress tensor, the body force, and the displacement respectively. If the coupled quasistatic theory is used to solve the thermal-elastic problem, the deformations (or stresses) and temperature distributions are determined simultaneously by solving Eqs. (2) and (3). When the coupling term [the last term in Eq. (2)] is neglected, the problem separates into two distinct equations to be solved consecutively: the first (heat conduction problem) to determine the temperature distribution, and the second (elasticity problem) to determine the deformations.

Now, consider a beam bounded by the planes $y = h$ and $y = +h$. The surface $y = -h$ is insulated, and the surface $y = +h$ is exposed to a constant heat input q for $t > 0$. The following assumptions are made.

1) The material of the beam is isotropic and obeys the restrictions of linear elasticity theory.

2) Mechanical and thermal properties remain unchanged with respect to temperature and time.

3) The plane-stress assumption holds; $\sigma_{zi} = 0$, $i = x, y$, and z.

4) For $L >> b$, $2h$, the semi-inverse assumption holds, i.e., $T = T(y)$, $\sqrt{xx} = \sigma_{xx}(y)$, and $\sigma_{yy} = \sigma_{xy} = 0$.

5) The dynamic effect on the thermal response is neglected (i.e., the inertia term in the equilibrium equation).

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The solution for the uncoupled quasistatic theory is available in any heat conduction textbook and is given as follows:

$$T^*(y^*,t^*) = t^* + \frac{1}{2}(y^{*2} + y^*) - \frac{1}{24} - \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2\pi^2} \cos n\pi(y^* + \frac{1}{2})e^{-n\pi t^*} \tag{4}$$

$$\sigma^*_{xx}(y^*,t^*) = -T^* + t^* + \frac{y^*}{2} + 48y^* \sum_{n=\text{even}}^{\infty} \frac{1}{n^4\pi^4} e^{-n\pi t^*} \tag{5}$$

where $T^* = \dfrac{k}{2hq}T$, $\sigma^*_{xx} = \dfrac{k}{2hq}\dfrac{\sigma_{xx}}{\alpha E}$

$t^* = \dfrac{\bar{k}}{4h^2}t$, and $y^* = \dfrac{y}{2h}$

When the coupled thermoelastic theory is used instead, the general thermoelastic problem will reduce to

$$k\frac{\partial^2 T}{\partial y^2} = \rho c_e \frac{\partial T}{\partial t} + \alpha(3\lambda + 2\mu)T_0 \frac{\partial}{\partial t}(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) \tag{6}$$

When Eq. (6) is expressed in terms of stresses, the two governing equations can be decoupled, and a series solution for T can be obtained satisfying the following initial and boundary conditions as well as the requirement of the zero stress resultants on the surfaces

$$T(y,0) = 0, \quad \frac{\partial T}{\partial y}(-h,t) = 0, \quad \text{and } k\frac{\partial T}{\partial y}(+h,t) = 0 \tag{7}$$

$$\int_{-h}^{h} \sigma_{xx} dy = 0, \qquad \int_{-h}^{h} \sigma_{xx} y dy = 0 \tag{8}$$

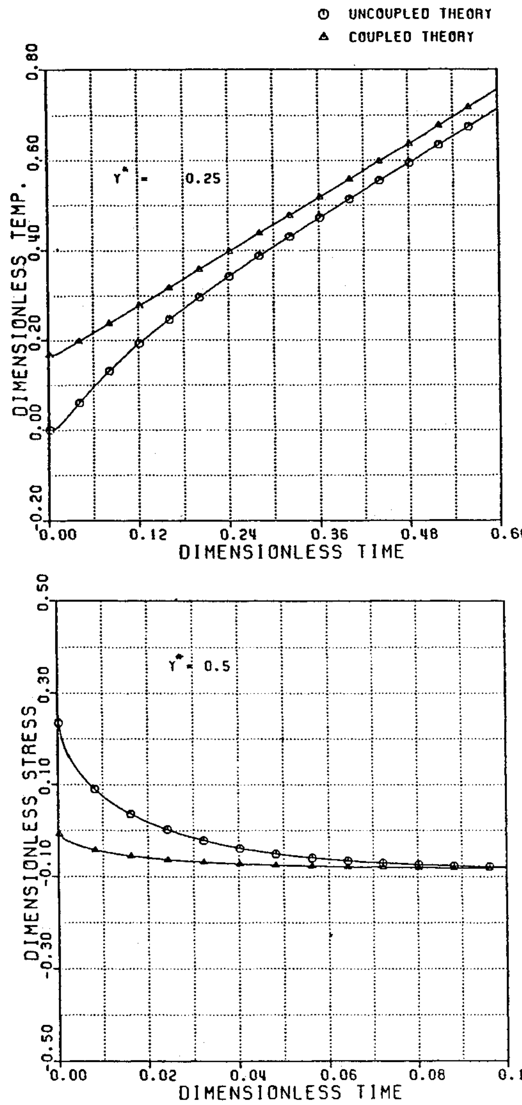

Fig. 1 The variation of stress and temperature with time.

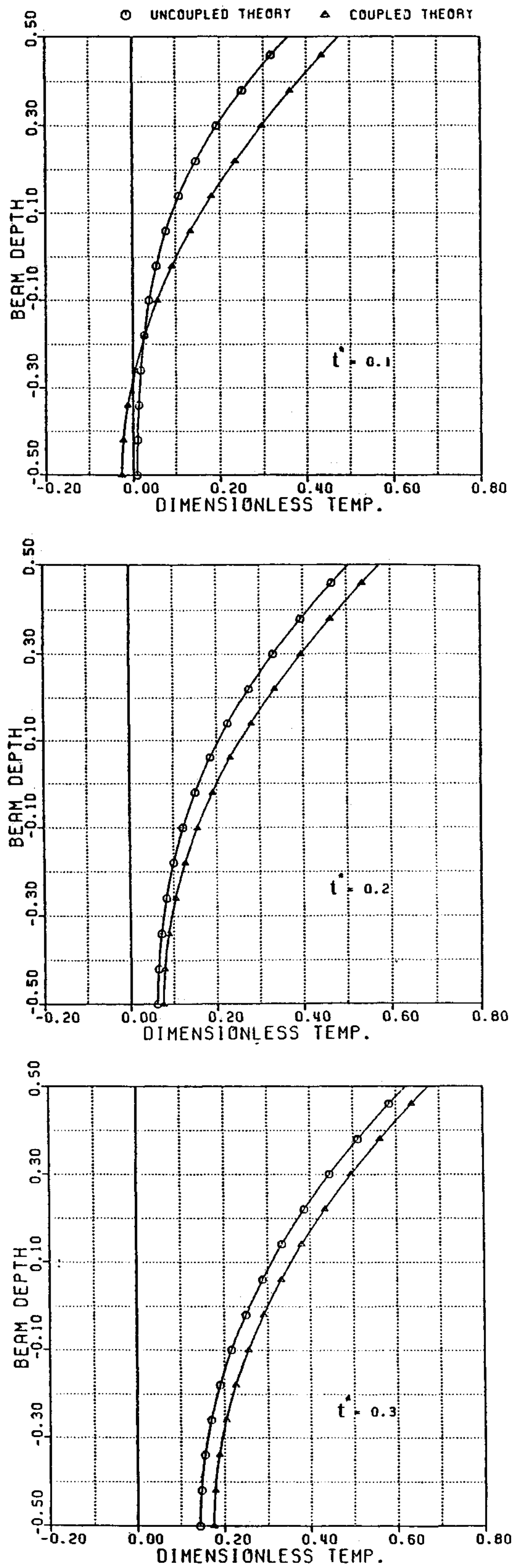

Fig. 2 Progress of temperature profiles.

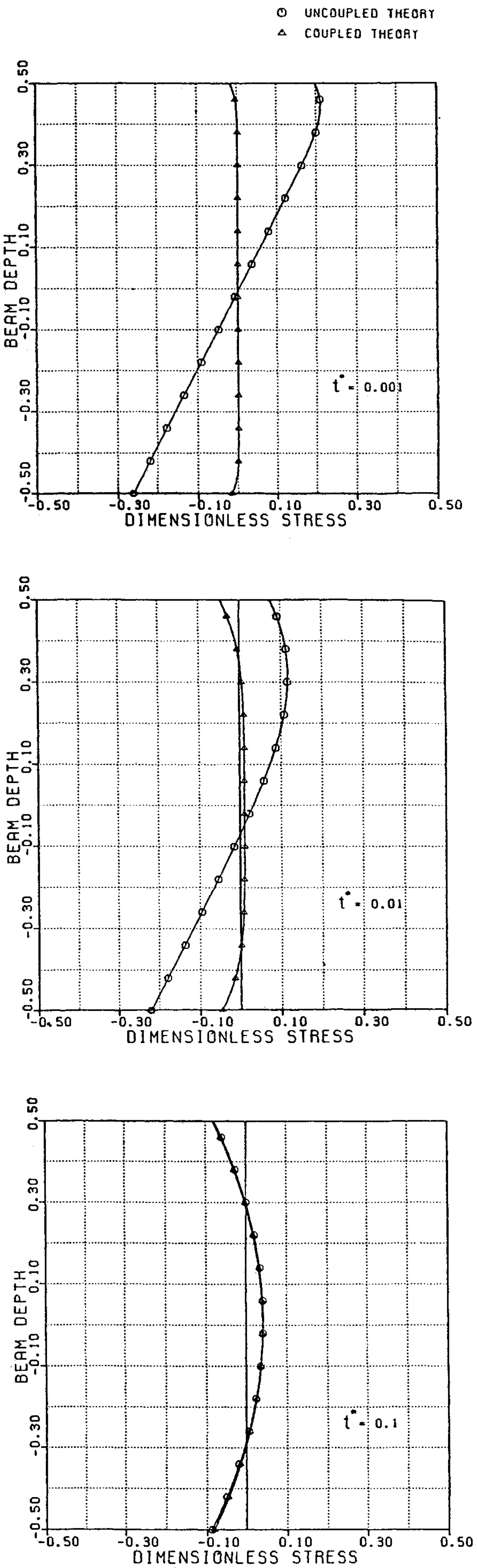

Fig. 3 Progress of stress profiles.

Then, the solution for the coupled problem will be as follows:

$$T^*(y^*,t^*) = At^*\frac{1}{2}(y^{*2}+y^*) - \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos 2n\pi y^* e^{-\gamma_n t^*} \tag{9}$$

$$\sigma^*_{xx}(y^*,t^*) = -T^* + At^* + \frac{y^*}{2} + \frac{1}{24} \tag{10}$$

where $A = \frac{\rho c_e}{a}$

$$\gamma_n = \frac{4n^2\pi^2\rho c_e}{\rho c_e + 2\alpha^2(1+\nu)(3\lambda+2\mu)T_0}$$

Discussion

If the reference temperature is taken to zero, A = 1 and $\gamma_n = 4\pi^2 n^2$, and consequently the results become material independent. Different comparisons between the coupled and the uncoupled theories for the stress and temperature have been made, and the following observations were made.

1) The uncoupled theory gives temperatures different than that of the coupled theory by up to 20%. The difference becomes constant as t^* approaches 0.36 (see Fig. 1).

2) It appears that the material reaches equilibrium at a time t^* between 0.2 and 0.3. After that time, the difference between the two theories is constant. Before that time, as time increases, the difference decreases, and the uncoupled theory underestimates the temperature (see Fig. 2).

3) Stress history and stress distribution over the beam depth for both theories coincide with each other as $t^* > 0.1$. Before that time, the uncoupled theory is more conservative than the coupled theory (see Figs. 1 and 3).

4) Stress distributions of the two theories approach each other faster than temperature distributions do. Stresses become identical as $t^* > 0.3$ (see Figs. 2 and 3).

Conclusions

Calculations on a simple heated beam illustrate the importance of properly accounting for thermomechanical coupling in computing the response of elastic solids subjected to abrupt heating. The calculations indicate that beams of elastic isotropic solids reach equilibrium faster than the use of uncoupled theory would predict. Furthermore, use of the uncoupled theory overestimates the thermal stresses present before equilibrium is reached. Consequently, the reasonableness of neglecting this coupling effect in thermoelasticity problems needs to be examined, and more attention needs to be given in cases of nonuniform thermal exposure and sudden high-intensity surface heating, such as due to fire or laser effects.

Acknowledgment

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References

[1]Mindlin, R. D., and Goodman, L. D., "Beam Vibrations with Time-Dependent Boundary Conditions," *Journal of Applied Mechanics*, Vol. 17, Dec. 1950, pp. 377-380.

[2]Boley, B. A., and Weiner, J. H., *Theory of Thermal Stresses*, John Wiley, New York, 1960.

[3]Soler, A. I., and Brull, M. A., "On the Solution to Transient Coupled Thermoelastic Problems by Perturbation Techniques," *Journal of Applied Mechanics*, Vol. 32, June 1965, pp. 389-399.

[4]Bahar, L. Y., and Hetnarski, R. B., "Coupled Thermoelasticity of a Layered Medium," *Journal of Thermal Stresses*, Vol. 3, April 1980, pp. 141-152.

[5]Takeuti, Y. and Tsuji, M., "Thermal Stress Problems in Industry, Part 6: Transient Thermal Stresses in a Plate due to Rolling with Considerations of Coupling Effect," *Journal of Thermal Stresses*, Vol. 5, Jan. 1982, pp. 53-65.

[6]Takeuti, Y., and Furkawa, T., "Some Considerations on Thermal Shock Problems in a Plate," *Journal of Applied Mechanics*, Vol. 48, March 1981, pp. 113-118.

[7]Takeuti, Y., and Furkawa, T., "Some Considerations on Thermal Shock Problems in a Plate," *Journal of Thermal Stresses*, Vol. 4, 1981, pp. 461-478.

[8]Chen, Y., Ghoneim, H., and Chen, Yu, "A Finite Element Solution of the Coupled Thermoelasticity Equations in A Slab," *Advances in Computer Technology, ASME*, Vol. 2, 1980, pp. 199-218.

[9]Li, Y. Y., Ghoneim, H., and Chen, Y., "A Numerical Method in Solving a Coupled Thermoelasticity Equation and Some Results," *Journal of Thermal Stresses*, Vol. 6, April 1983, pp. 253-280.

Stiffness-Matrix Condition Number and Shape Sensitivity Errors

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Introduction

FOR static response, the condition number of the stiffness matrix is an upper bound to the amplification of errors in structural properties and loads. However, even though typical stiffness matrices have condition numbers larger than one million, we do not expect that errors or variations in the structure or loads would be amplified so much. The present Note seeks to explain why in most cases our expectation is fulfilled. It also presents an example of a case associated with shape sensitivity analysis where the worst-case scenario predicted by the condition number is much closer to the actual error amplification. A criterion is proposed that is closer to the actual error magnification than the condition number.

Consider the discretized equations of equilibrium of static response such as those generated by a finite-element model

$$Ku = f \tag{1}$$

where K is the $n \times n$ symmetric, positive, definite, stiffness matrix, u the displacement vector, and f the load vector. The condition number of K, $c(K)$ is defined as

$$c(K) = \|K\| \; \|K^{-1}\| \tag{2}$$

when the 2-norm is used

$$c(K) = \lambda_n / \lambda_1 \tag{3}$$

where λ_i denotes the ith eigenvalue of K. It is well known (e.g., see Ref. 1) that $c(K)$ is an upper bound on the sensitivity of u to perturbations in K and f. That is, if we perturb f by Δf then

$$\frac{\|\Delta u\|}{\|u\|} \leq c(K) \frac{\|\Delta f\|}{\|f\|} \tag{4}$$

and if we perturb K by ΔK then

$$\frac{\|\Delta u\|}{\|u + \Delta u\|} \leq c(K) \frac{\|\Delta K\|}{\|K\|} \tag{5}$$

The condition number for most stiffness matrices generated by finite-element models runs into the millions. This would appear to indicate that the computed displacement field can be extremely sensitive to small errors in the stiffness matrix and force vectors. In spite of this theoretical sensitivity, we continue to approximate the stiffness matrix (e.g., by reduced integration) and the force vector (e.g., lumping loads) without fear of the huge amplification of errors predicted by the condition number. It is known, in fact, that the condition number may be an overly conservative estimate of error sensitivity.[2]

The condition number is particularly overconservative for predicting sensitivity to changes in the load vector. For a given K and f, it is always possible to find a ΔK to make Eq. (5) an equality. Also, $c(K)$ can be a good predictor of roundoff error amplification so that if the condition number is 10^7 and we work with 7-digit numbers, the errors in u can be very large. In the following it is assumed that the number of digits available for computation is much larger than the condition number (a typical case in finite-element computation is $c(K) = 10^7$ with 15-digit computations). However, it is not usually possible to find a Δf to make Eq. (4) an equality.[2] The present Note derives a sharper estimate for sensitivity to load errors. It also presents a case where the extreme sensitivity predicted by the condition number is more closely realized.

Error Analysis

Let the eigenvectors of K be denoted as u_i, $i = 1,\ldots,n$ normalized to $\|u_i\| = 1$ with λ_i being the corresponding eigenvalues. We expand the load vector in terms of the eigenvectors as

$$f = \sum_{i=1}^{n} \alpha_i u_i \tag{6}$$

and similarly the perturbation or error in the load as

$$\Delta f = \sum_{i=1}^{n} \Delta\alpha_i u_i \tag{7}$$

It is then easy to check that u can be obtained as

$$u = \sum_{i=1}^{n} (\alpha_i / \lambda_i) u_i \tag{8}$$

with a similar expansion for Δu. The error amplification factor e is defined as

$$e = \frac{\|\Delta u\|}{\|\Delta u\|} \Big/ \frac{\|\Delta f\|}{\|f\|} \tag{9}$$

Using the orthonormality of the eigenvectors we get

$$e^2 = \frac{(\Sigma_{i=1}^{n} \Delta\alpha_i^2 / \lambda_i^2)(\Sigma_{i=1}^{n} \alpha_i^2)}{(\Sigma_{i=1}^{n} \alpha_i^2 / \lambda_i^2)(\Sigma_{i=1}^{n} \Delta\alpha_i^2)} \tag{10}$$

It is easy to check that the worst case is when the perturbation is in the shape of the first eigenvector, $\Delta f = \Delta\alpha_1 u_1$ so that an upper bound on e, called here the error magnification index

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